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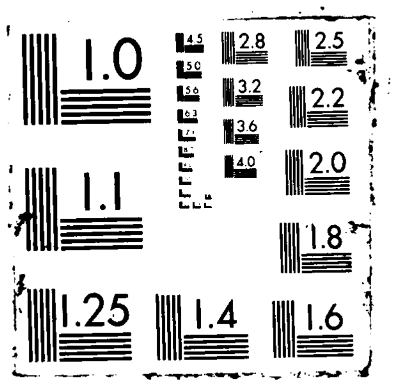
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ERGODIC PROPERTIES OF STATIONARY STABLE PROCESSES

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We derive spectral necessary and sufficient conditions for stationary symmetric stable processes to be metrically transitive and mixing. We then consider some important classes of stationary stable processes: Sub-Gaussian stationary processes and stationary stable processes with a harmonic spectral representation are never metrically transitive, the latter in sharp contrast with the Gaussian case. Stable processes with a harmonic spectral representation satisfy a strong law of large numbers even though they are not generally stationary. For doubly stationary stable processes, sufficient conditions are derived for metric transitivity and mixing, and necessary and sufficient conditions for a strong law of large numbers.

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stable processes * ergodic theory * stationary processes * spectral representations

1. Introduction

Stationary symmetric α-stable (SαS) processes have been characterized in [12] and form a richer, and therefore more unyielding class of processes than the stationary Gaussian processes. For instance, while all stationary Gaussian processes which are continuous in probability have a harmonic spectral representation, this is not so in the stable case; and when $1<\alpha<2$ the class of SαS moving averages is disjoint from the class of regular SαS processes with a harmonic representation, whereas in the Gaussian case, these two classes coincide (cf. [5]).

Using their description developed in [12], we derive necessary and sufficient conditions for stationary SαS processes to be metrically transitive (Theorem 1) and mixing (Theorem 2). We then consider some important special classes of stationary SαS processes. We show that sub-Gaussian stationary processes are never metrically transitive (Theorem 3). SαS moving averages are of course mixing, and stationary

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$S\alpha S$ solutions of linear, constant coefficient, stable stochastic differential equations are strongly mixing; the latter is the continuous time analog of a result in [13] for discrete time autoregressive $S\alpha S$ processes and is established likewise. Stationary $S\alpha S$ processes with a harmonic spectral representation are never metrically transitive (Theorem 4), in sharp contrast with the Gaussian case. Also $S\alpha S$ processes with a harmonic spectral representation satisfy a strong law of large numbers (Theorem 5) even though they are not generally stationary; this is an L_α analog of results in [10] for L_2-stationary processes. Finally in Section 6 we introduce doubly stationary $S\alpha S$ processes—a new class of $S\alpha S$ stationary processes with "stationary" spectral representations which includes Gaussian, α-sub-Gaussian and $S\alpha S$ moving average processes—and give sufficient conditions for metric transitivity (Theorem 6) and mixing (Theorem 7), as well as necessary and sufficient conditions for them to satisfy the strong law of large numbers (Theorem 8).

We concentrate on real processes defined on the real line, but similar results hold for real sequences, as well as real processes defined on certain more general groups (see e.g. [21] where ergodic properties for harmonizable $S\alpha S$ processes on LCA groups are discussed). The assumption that the process is real is needed when considering metric transitivity and mixing, because of the use of the dense set of trigonometric polynomials (cf. [19, p. 163]), but is of no significance when considering laws of large numbers.

We now introduce some basic notation and properties used throughout the paper. A real random variable Y is $S\alpha S$, $0<\alpha\le 2$, if $E\exp(irY)=\exp(-c_Y|r|^\alpha)$ for all r and some $c_Y\ge 0$. A process $X=\{X_t;\ -\infty<t<\infty\}$ is $S\alpha S$ if all finite linear combinations $\sum a_iX_{t_i}$ are $S\alpha S$. For a $S\alpha S$ random variable Y, set $\|Y\|_\alpha=c_Y^{1/\alpha}$. Then $\|\cdot\|_\alpha^{1\wedge\alpha}$ defines a norm in the case $1\le\alpha\le 2$, and a quasi-norm in the case $0<\alpha<1$, on the linear span of the $S\alpha S$ process X, $\mathcal{L}(X)$, which metrizes convergence in probability. Also, for $0<p<\alpha$,

$$(E|Y|^p)^{1/p}=C(p,\alpha)\|Y\|_\alpha$$

where the constant $C(p,\alpha)$ depends only on p, α and not on Y [22]. Stationary $S\alpha S$ processes X with $0<\alpha<2$ have finite dimensional characteristic functions of the form

$$E\exp\left\{i\sum_{n=1}^N a_nX_{t_n}\right\}=\exp\left\{-\left\|\left(\sum_{n=1}^N a_nU_{t_n}\right)\phi\right\|_\alpha^\alpha\right\} \tag{1}$$

and thus the following spectral representation in law

$$\{X_t,\ -\infty<t<\infty\}\overset{\mathcal{L}}{=}\left\{\int_E (U_t\phi)(\lambda)\,\mathrm{d}Z(\lambda),\ -\infty<t<\infty\right\} \tag{2}$$

[12]. Here (E,Σ,μ) is a measure space, $\phi\in L_\alpha(E,\Sigma,\mu)\triangleq L_\alpha(\mu)$, $\{U_t,\ -\infty<t<\infty\}$ is a group of isometries on $L_\alpha(\mu)$, and Z is the canonical independently scattered $S\alpha S$ measure on (E,Σ,μ); i.e. for all disjoint sets $E_1,\dots,E_n\in\Sigma$ of finite μ-measure, $Z(E_1),\dots,Z(E_n)$ are independent with $E\exp\{irZ(E_k)\}=\exp\{-|r|^\alpha\mu(E_k)\}$, so that,

for all $f \in L_\alpha(\mu)$,

$$E \exp\left\{ i \int f \, dZ \right\} = \exp\{-\|f\|_\alpha^\alpha\}.$$

Denoting by $\mathscr{F}_X = \sigma\{X_t, -\infty < t < \infty\}$ the σ-algebra of X-measurable events, the shift transformation associated with the stationary SαS process is defined in the usual way for all X-measurable (i.e. $\mathscr{F}_X$-measurable) events and random variables, so that e.g. $g(X_{t_1}, \dots, X_{t_n})$ shifted by τ becomes $g(X_{t_1+\tau}, \dots, X_{t_n+\tau})$ (cf. [19]). For the notions of metric transitivity and mixing, and for laws of large numbers, it is necessary that the process $\{\eta_\tau, -\infty < \tau < \infty\}$ obtained by shifting an X-measurable random variable η be measurable. This is the case if the original strationary SαS process X is measurable, or has a measurable modification (cf. [19]). In view of the following property we assume without further notice that the group $\{U_t\}$ is strongly measurable on all of $L^\alpha(\mu)$ and that μ is σ-finite.

Theorem 0. *For a stationary* SαS *process* X *with spectral representation* (2) *the following are equivalent*:

(i) X *has a measurable modification,*

(ii) X *is continuous in probability,*

(iii) $\{U_t\}$ *is strongly measurable on* $F \triangleq \overline{\mathrm{sp}}\{U_t\phi\}_{L_\alpha(\mu)}$,

(iv) $\{U_t\}$ *is strongly continuous on* F.

Proof. By [6], X has a measurable modification if and only if the map $L : \mathbb{R} \to L_\alpha(\mu)$ given by $L(t) = U_t\phi$ is measurable, since the (quasi-) norm $\|\cdot\|_\alpha^{1 \wedge \alpha}$ on $\mathscr{S}(X)$ metrizes convergence in probability, and by (1) the linear extension of the map $X_t \mapsto U_t\phi$ is an isometry of $\mathscr{S}(X)$ into $L_\alpha(\mu)$. If L is measurable, its range is separable, and we may thus assume without loss of generality that (E, Σ, μ) is σ-finite. More significantly, measurability of L implies measurability of the map $t \mapsto U_tf$ for each $f \in F$, i.e. strong measurability of the group $\{U_t\}$ on F. This, however, implies the strong continuity of $\{U_t\}$ on F (see [8, p. 616]), and hence that X is continuous in probability (since $X_t \to X_{t_0}$ in probability if and only if $\|U_t\phi - U_{t_0}\phi\|_{L_\alpha(\mu)} \to 0$). Thus (i)$\Rightarrow$(iii)$\Rightarrow$(iv)$\Rightarrow$(ii) and the proof is complete by the well known property (ii)$\Rightarrow$(i). $\square$

That (i) implies (ii) when $\alpha = 2$ (in fact for all weakly stationary processes with finite second moment) was shown in [7].

2. Metric transitivity

A stationary process X is called metrically transitive or ergodic if any of the following equivalent conditions is satisfied (cf. [9]): (i) the shift invariant measurable sets $\mathcal{J}$ of X have probability zero or one; (ii) for each X-measurable random variable η with $E|\eta|<\infty$,

$$\lim_{T\to\infty}\frac{1}{T}\int_0^T \eta_\tau\,\mathrm{d}\tau = E\eta \quad \text{a.s.} \tag{3}$$

where η_τ is η shifted by τ; (iii) whenever $A\in\sigma(X_t, t\leq 0)$, $B\in\sigma(X_t, t\geq 0)$ and B_τ is the event B shifted by τ,

$$\lim_{T\to\infty}\frac{1}{T}\int_0^T P(A\cap B_\tau)\,\mathrm{d}\tau = P(A)P(B) \quad \text{a.s.} \tag{4}$$

By a result of Maruyama and Granander [19, 9], a stationary Gaussian process is metrically transitive if and only if its spectral measure has no atoms. For general stationary stable processes, we have the following characterization.

Theorem 1. *A stationary* SαS *process* X *with* $0<\alpha\leq 2$ *and spectral representation* (2) *is metrically transitive if and only if for each* $h\in\overline{\mathrm{sp}}\{U_t\phi, -\infty<t<\infty\}_{L^\alpha(\mu)}$,

$$\lim_{T\to\infty}\frac{1}{T}\int_0^T \|U_\tau h - h\|_\alpha^\alpha\,\mathrm{d}\tau = 2\|h\|_\alpha^\alpha \tag{5}$$

and

$$\lim_{T\to\infty}\frac{1}{T}\int_0^T \|U_\tau h - h\|_\alpha^{2\alpha}\,\mathrm{d}\tau = 4\|h\|_\alpha^{2\alpha}. \tag{6}$$

Proof. As in the standard proof for Gaussian processes, it suffices to have (3) for r.v.'s η of the form $\eta=\exp[\mathrm{i}\sum_{n=1}^N a_n X_{t_n}]$, and this is where the fact that X is real is used (see [19] or [9]). Then, putting $h=(\sum_{n=1}^N a_n U_{t_n})\phi$, we have $\eta_\tau = \exp[\mathrm{i}\int_E U_\tau h\,\mathrm{d}Z]$ and

$$Y_T \triangleq \frac{1}{T}\int_0^T \eta_\tau\,\mathrm{d}\tau = \frac{1}{T}\int_0^T \exp\left[\mathrm{i}\int_E U_\tau h\,\mathrm{d}Z\right]\mathrm{d}\tau.$$

By Birkhoff's theorem [19], $Y_T\to E(\eta\,|\,\mathcal{J})\triangleq Y_\infty$ a.s. Thus (3) is satisfied, i.e. $Y_\infty = E\eta$, if and only if $E|Y_\infty|^2=|EY_\infty|^2$, if and only if $\lim_T E|Y_T|^2 = \lim_T|EY_T|^2$. But

$$EY_T = \frac{1}{T}\int_0^T \exp[-\|U_\tau h\|_\alpha^\alpha]\,\mathrm{d}\tau = \exp[-\|h\|_\alpha^\alpha],$$

and

$$E|Y_T|^2 = \frac{1}{T^2}\int_0^T\int_0^T \exp[-\|(U_\tau - U_\sigma)h\|_\alpha^\alpha]\,\mathrm{d}\tau\,\mathrm{d}\sigma.$$

Since $|x|^\alpha+|y|^\alpha-|x-y|^\alpha$ is a positive definite function of x and y, $|(U_\tau h)(\lambda)|^\alpha + |(U_\sigma h)(\lambda)|^\alpha - |[(U_\tau - U_\sigma)h](\lambda)|^\alpha$ is a positive definite function of τ and σ for each λ, and thus so is its μ-integral over E: $2\|h\|_\alpha^\alpha - \|(U_\tau - U_\sigma)h\|_\alpha^\alpha$. Since the latter depends only on the difference $\tau-\sigma$, and is continuous, we have by Bochner's theorem

$$2\|h\|_\alpha^\alpha - \|(U_\tau - U_\sigma)h\|_\alpha^\alpha = \int_{-\infty}^{\infty} e^{i(\tau-\sigma)u}\, d\nu(u)$$

where ν is a finite symmetric measure. Then proceeding as in [9, p. 77], we obtain

$$\begin{aligned}\frac{E|Y_T|^2}{|EY_T|^2} &= \frac{1}{T^2}\int_0^T\int_0^T \exp\left[\int_{-\infty}^{\infty} e^{i(\tau-\sigma)u}\, d\nu(u)\right] d\tau\, d\sigma \\ &= 1 + \sum_{k=1}^{\infty}\frac{1}{k!}\frac{1}{T^2}\int_0^T\int_0^T \left\{\int_{-\infty}^{\infty} e^{i(\tau-\sigma)u}\, d\nu^{(k)}(u)\right\} d\tau\, d\sigma \\ &\xrightarrow[T\to\infty]{} 1 + \sum_{k=1}^{\infty}\frac{1}{k!}\nu^{(k)}\{0\}\end{aligned}$$

where $\nu^{(k)}$ is the k-fold convolution of ν. It follows that X is metrically transitive if and only $\nu^{(k)}\{0\} = 0$ for all $k \geq 1$ (and all $h \in \overline{\mathrm{sp}}\{U_t\phi,\ -\infty < t < \infty\} \subset L_\alpha(\mu)$).

Since the function $\int_{-\infty}^{\infty} e^{i\tau u}\, d\nu(u) = 2\|h\|_\alpha^\alpha - \|U_\tau h - h\|_\alpha^\alpha$ is even, we have by the inversion formula

$$\nu\{0\} = 2\|h\|_\alpha^\alpha - \lim_{T\to\infty}\frac{1}{T}\int_0^T \|U_\tau h - h\|_\alpha^\alpha\, d\tau$$

and thus $\nu\{0\} = 0$ if and only if (5) is satisfied. Also

$$\begin{aligned}\nu^{(2)}\{0\} &= \int_{-\infty}^{\infty}\nu\{-x\}\, d\nu(x) = \sum_x \nu\{-x\}\nu\{x\} = \sum_x \nu^2\{x\} \qquad (\nu \text{ is symmetric}) \\ &= \lim_{T\to\infty}\frac{1}{T}\int_0^T (2\|h\|_\alpha^\alpha - \|U_\tau h - h\|_\alpha^\alpha)^2\, d\tau \qquad (\text{by Wiener's theorem [15]}) \\ &= 4\|h\|_\alpha^{2\alpha} - 4\|h\|_\alpha^\alpha \lim_{T\to\infty}\frac{1}{T}\int_0^T \|U_\tau h - h\|_\alpha^\alpha\, d\tau + \lim_{T\to\infty}\frac{1}{T}\int_0^T \|U_\tau h - h\|_\alpha^{2\alpha}\, d\tau\end{aligned}$$

from which it follows that $\nu^{(k)}\{0\} = 0$, $k = 1, 2$, if and only if (5) and (6) are satisfied. The proof is completed by noting that (from the above calculation) $\nu^{(2)}\{0\} = 0$ implies ν has no atoms and thus $\nu^{(k)}\{0\} = 0$ for all $k \geq 2$.

When a stationary SαS process X is metrically transitive we can use Birkhoff's theorem to estimate its covariation function which plays a role analogous to that of the covariance when $\alpha = 2$ [4]. Indeed when $1 < p < \alpha < 2$ we have

$$\frac{1}{T}\int_0^T X_t X_{t+\tau}^{\langle p-1\rangle}\, dt \xrightarrow[T\to\infty]{\text{a.s.}} E\{X_0 X_\tau^{\langle p-1\rangle}\} = C^p(p,\alpha)\frac{\mathrm{Cov}(X_0, X_\tau)}{\|X_0\|_\alpha^{\alpha-p}},$$

where $x^{\langle\beta\rangle} = |x|^\beta \operatorname{sign}(x)$ and the equality follows from [3]. For $\tau = 0$ this gives the scaling constant of the process:

$$\frac{1}{T}\int_0^T |X_t|^p\, dt \xrightarrow[T\to\infty]{\text{a.s.}} E|X_0|^p = C^p(p,\alpha)\|X_0\|_\alpha^p.$$

A-1 20

3. Mixing

A stationary process is called mixing if either of the following equivalent conditions is satisfied: (i) with A, B, B_T as in (4),

$$\lim_{T\to\infty} P(A \cap B_T) = P(A)P(B), \tag{7}$$

(ii) whenever ξ is $(X_t, t \le 0)$-measurable, η is $(X_t, t \ge 0)$-measurable, $E|\xi| < \infty$, $E|\eta| < \infty$, and η_T is η shifted by T, we have

$$\lim_{T\to\infty} E(\xi\eta_T) = E\xi \cdot E\eta. \tag{8}$$

It is clear that mixing is a stronger property than metric transitivity. A stationary Gaussian process with harmonic spectral representation $X_t = \operatorname{Re}\int_{-\infty}^{\infty} e^{it\lambda}\, dW(\lambda)$ is mixing if and only if its covariance $R(T) = \int_{-\infty}^{\infty} e^{iT\lambda}\, d\mu(\lambda)$ tends to zero as $T \to \infty$. For non-Gaussian stationary stable processes, we have the following characterization.

Theorem 2. *A stationary SαS process X with $0 < \alpha < 2$ and spectral representation (2) is mixing if and only if for every $g \in \operatorname{sp}\{U_t\phi, t \le 0\} \subset L^\alpha(\mu)$ and $h \in \operatorname{sp}\{U_t\phi, t \ge 0\} \subset L^\alpha(\mu)$,*

$$\lim_{T\to\infty} \|g + U_T h\|_\alpha^\alpha = \|g\|_\alpha^\alpha + \|h\|_\alpha^\alpha. \tag{9}$$

Proof. It suffices to have (8) for r.v.'s of the form $\eta = \exp[i\sum_{n=1}^{N} a_n X_{t_n}]$, $t_n \ge 0$, $\xi = \exp[i\sum_{m=1}^{M} b_m X_{s_m}]$, $s_m \le 0$. Putting $h = \sum_{n=1}^{N} a_n U_{t_n}\phi$, $g = \sum_{m=1}^{M} b_m U_{s_m}\phi$, we have

$$E\xi = E\exp\left[i\int_S g\, dZ\right] = \exp[-\|g\|_\alpha^\alpha],$$

$$E\eta_T = E\exp\left[i\int_S U_T h\, dZ\right] = \exp[-\|U_T h\|_\alpha^\alpha] = \exp[-\|h\|_\alpha^\alpha],$$

$$E(\xi\eta_T) = E\exp\left[i\int_S (g + U_T h)\, dZ\right] = \exp[-\|g + U_T h\|_\alpha^\alpha],$$

from which (9) follows.

4. Sub-Gaussian processes

A process X is called α-sub-Gaussian if its finite dimensional characteristic functions are of the form

$$E\exp\left[i\sum_{n=1}^{N} a_n X_{t_n}\right] = \exp\left[-\left(\frac{1}{2}\sum_{m,n=1}^{N} a_m a_n R(t_m - t_n)\right)^{\alpha/2}\right]$$

where $R(t, s)$ is a positive definite function, or equivalently, if $X_t = A^{1/2}G_t$, $-\infty < t < \infty$, where A is positive $\alpha/2$-stable and independent of the Gaussian process G which has mean zero and covariance function R. We show that stationary sub-Gaussian processes are not ergodic.

Theorem 3. *Sub-Gaussian stationary processes are never metrically transitive.*

Proof. Since for a zero mean normal r.v. ξ we have $E|\xi|^\alpha = D_\alpha(E\xi^2)^{\alpha/2}$, where the constant D_α does not depend on ξ, it follows that

$$E\exp\left\{i\sum_{n=1}^{N} a_n X_{t_n}\right\} = \exp\left\{-\left[\tfrac{1}{2}E\left(\sum_{n=1}^{N} a_n G_{t_n}\right)^2\right]^{\alpha/2}\right\}$$

$$= \exp\left\{-2^{-\alpha/2} D_\alpha^{-1} E\left|\sum_{n=1}^{N} a_n G_{t_n}\right|^\alpha\right\}.$$

Hence

$$\{X_t, -\infty < t < \infty\} \overset{d}{=} \left\{2^{-1/2} D_\alpha^{-1/\alpha} \int_\Omega U_t G_0 \, dZ, -\infty < t < \infty\right\}$$

where $G_t = U_t G_0$, and Z is the canonical independently scattered SαS measure on $(\Omega, \mathcal{F}, P)$. Since X is stationary, so is G. Checking condition (5) with $h = G_0$, we have

$$\frac{1}{T}\int_0^T \|U_\tau h - h\|_\alpha^\alpha \, d\tau = \frac{1}{T}\int_0^T E|G_\tau - G_0|^\alpha \, d\tau = \frac{D_\alpha}{T}\int_0^T (E|G_\tau - G_0|^2)^{\alpha/2} \, d\tau$$

$$= \frac{D_\alpha}{T}\int_0^T (2[R(0) - R(\tau)])^{\alpha/2} \, d\tau$$

$$= D_\alpha [R(0)]^{\alpha/2} \frac{1}{T}\int_0^T \left(2\left[1 - \frac{R(\tau)}{R(0)}\right]\right)^{\alpha/2} d\tau$$

$$\le \|h\|_\alpha^\alpha \left(\frac{1}{T}\int_0^T 2\left[1 - \frac{R(\tau)}{R(0)}\right] d\tau\right)^{\alpha/2} \qquad \text{(Jensen)}$$

$$\xrightarrow[T \to \infty]{} \|h\|_\alpha^\alpha \left(2\left[1 - \frac{\mu\{0\}}{R(0)}\right]\right)^{\alpha/2} < 2\|h\|_\alpha^\alpha$$

where $R(\tau) = \int_{-\infty}^{\infty} e^{i\tau\lambda} \, d\mu(\lambda)$, and the inequality is strict for $0 < \alpha < 2$ even when $\mu\{0\} = 0$. Hence condition (5) is not satisfied and X cannot be ergodic.

The ergodic decomposition of a sub-Gaussian process X can be easily described in terms of the ergodic decomposition of the corresponding Gaussian process G (which may or may not be metrically transitive) using the fact that, modulo null sets, the σ-field of invariant sets of X equals the smallest σ-field containing the σ-field generated by the $\alpha/2$-stable r.v. A and the σ-field of invariant sets of Y.

5. Fourier transforms

We say that a complex SαS process X has a harmonic spectral representation if

$$X_t = \int_{-\infty}^{\infty} e^{it\lambda}\, dW(\lambda), \quad -\infty < t < \infty,$$

where W is a complex independently scattered SαS measure on $(\mathbb{R}^1, \mathscr{B}^1, \mu)$, μ finite. Then for every complex $f \in L_\alpha(\mu)$,

$$E \exp\left\{ i \operatorname{Re}\left[\int_{-\infty}^{\infty} f\, dW \right] \right\} = \exp\left\{ -\int_{-\infty}^{\infty} \int_{-\pi}^{\pi} |\operatorname{Re}[f(\lambda)\, e^{i\theta}]|^\alpha \, d\nu(\lambda, \theta) \right\} \tag{10}$$

where ν is a measure on the Borel subsets of $\mathbb{R}^1 \times (-\pi, \pi]$ with a marginal $\mu : \nu\{B \times (-\pi, \pi]\} = \mu(B)$ [14, 2]. X is stationary if and only if the measure W is rotationally invariant, i.e. the distribution of the process $\{e^{i\phi} W(B), B \in \mathscr{B}^1\}$ does not depend on ϕ, in which case $\nu = \mu \times (\text{Leb}/2\pi)$ and for $f \in L_\alpha(\mu)$ we have

$$E \exp\left\{ i \operatorname{Re}\left[\int_{-\infty}^{\infty} f\, dW \right] \right\} = \exp\left\{ -C_\alpha \int_{-\infty}^{\infty} |f|^\alpha \, d\mu \right\}$$

where $C_\alpha = (2\pi)^{-1} \int_{-\pi}^{\pi} |\cos\theta|^\alpha \, d\theta$. Unlike the Gaussian case $\alpha = 2$, where all stationary Gaussian processes which are continuous in probability have a harmonic representation, there are stationary SαS processes with $0 < \alpha < 2$ which are continuous in probability but do not have a harmonic representation, such as sub-Gaussian processes and moving averages of SαS processes with stationary independent increments [5].

For a real (stationary) SαS process X we say that it has a harmonic representation if

$$X_t = \operatorname{Re} \int_{-\infty}^{\infty} e^{it\lambda}\, dW(\lambda)$$

where W is as above (and is rotationally invariant). We show that when $0 < \alpha < 2$ such processes are never ergodic, in sharp contrast with the the stationary Gaussian processes ($\alpha = 2$) which are ergodic if and only if the spectral measure μ has no atoms. This has also been indicated in [17]. Even though these processes are not ergodic, their spectral measure μ can be estimated consistently under the usual assumptions [18].

Theorem 4. *A real stationary* SαS *process with a harmonic representation is never metrically transitive when* $0<\alpha<2$.

Proof. With some minor adjustments in the proof of Theorem 1, we have that X is metrically transitive if and only if

$$\begin{aligned} &\frac{1}{T}\int_0^T \|(\mathrm{e}^{\mathrm{i}\tau\lambda}-1)h(\lambda)\|_\alpha^\alpha \,\mathrm{d}\tau \to 2\|h\|_\alpha^\alpha, \\ &\frac{1}{T}\int_0^T \|(\mathrm{e}^{\mathrm{i}\tau\lambda}-1)h(\lambda)\|_\alpha^{2\alpha} \,\mathrm{d}\tau \to 4\|h\|_\alpha^{2\alpha}, \end{aligned} \tag{11}$$

for all complex $h\in L_\alpha(\mu)$. But

$$\begin{aligned} \frac{1}{T}\int_0^T \|(\mathrm{e}^{\mathrm{i}\tau\lambda}-1)h(\lambda)\|_\alpha^\alpha \,\mathrm{d}\tau &= \int_{-\infty}^{\infty} |h(\lambda)|^\alpha \left\{\frac{1}{T}\int_0^T \left|2\sin\frac{\tau\lambda}{2}\right|^\alpha \mathrm{d}\tau\right\} \mathrm{d}\mu(\lambda) \\ &= \int_{\lambda\neq 0} |h(\lambda)|^\alpha \left\{\frac{2}{T\lambda}\int_0^{T\lambda/2} |2\sin u|^\alpha \,\mathrm{d}u\right\} \mathrm{d}\mu(\lambda) \\ &\xrightarrow[T\to\infty]{} \frac{1}{\pi}\int_0^\pi |2\sin u|^\alpha \,\mathrm{d}u \cdot \int_{\lambda\neq 0} |h|^\alpha \,\mathrm{d}\mu \\ &\triangleq D_\alpha \int_{\lambda\neq 0} |h|^\alpha \,\mathrm{d}\mu. \end{aligned}$$

Note that when $\alpha=2$, $D_2=2$ and thus (11) is satisfied provided $\mu\{0\}=0$. We now show that when $0<\alpha<2$, $D_\alpha<2$, and thus (11) is not satisfied and X is not metrically transitive. Indeed, by Jensen's inequality we have

$$\begin{aligned} D_\alpha &= \frac{1}{\pi}\int_0^\pi |2\sin u|^\alpha \,\mathrm{d}u \\ &= \frac{1}{\pi}\int_0^\pi (|2\sin u|^2)^{\alpha/2} \,\mathrm{d}u \le \left(\frac{1}{\pi}\int_0^\pi |2\sin u|^2 \,\mathrm{d}u\right)^{\alpha/2} \\ &= 2^{\alpha/2}. \quad \square \end{aligned}$$

We now turn our attention to laws of large numbers (LLN). We consider complex processes from which the results for their real parts follow immediately. Let X be a complex SαS process with a harmonic representation. It is easily seen that

$$\begin{aligned} \frac{1}{T}\int_0^T X_t \,\mathrm{d}t &= \int_{-\infty}^{\infty} \left\{\frac{1}{T}\int_0^T \mathrm{e}^{\mathrm{i}t\lambda} \,\mathrm{d}t\right\} \mathrm{d}W(\lambda) \\ &\xrightarrow[T\to\infty]{} \int_{-\infty}^{\infty} 1_{\{0\}}(\lambda)\,\mathrm{d}W(\lambda) = W\{0\} \end{aligned}$$

in probability. Thus X satisfies a weak LLN if and only if $W\{0\}=0$. When X is stationary (i.e. W is rotationally invariant) and $1<\alpha\le 2$, then by Birkhoff's theorem the above convergence is also a.s., and X satisfies a strong LLN (SLLN) if and only if $\mu\{0\}=0$. Following the approach in [10], where L_2-stationary processes are considered, we show that this latter property remains true even when X is not stationary.

Theorem 5. *Let X be a complex* SαS *process with harmonic representation and $1<\alpha\leq 2$. Then*

$$\frac{1}{T}\int_0^T X_t\,\mathrm{d}t \xrightarrow[T\to\infty]{} W\{0\} \quad a.s.$$

and X satisfies a SLLN *if and only if $\mu\{0\}=0$.*

Proof. The proof parallels that of theorems 1′ and 2′ in [10] as outlined on pp. 303–304. Here we only point out the main adjustments necessary when $1<\alpha<2$. The first step is to show that it suffices to establish the a.s. convergence along the integers since

$$Z_k=\sup_{k\leq T\leq k+1}\left|\frac{1}{T}\int_0^T X_t\,\mathrm{d}t-\frac{1}{k}\int_0^k X_t\,\mathrm{d}t\right| \xrightarrow[k\to\infty]{} 0.$$

Indeed from

$$Z_k\leq\frac{1}{k+1}\frac{1}{k}\int_0^k |X_t|\,\mathrm{d}t+\frac{1}{k}\int_k^{k+1}|X_t|\,\mathrm{d}t$$

we obtain, for $1<p<\alpha$,

$$\{EZ_k^p\}^{1/p}\leq\frac{1}{k+1}\left\{E\frac{1}{k}\int_0^k |X_t|^p\,\mathrm{d}t\right\}^{1/p}+\frac{1}{k}\left\{E\int_k^{k+1}|X_t|^p\,\mathrm{d}t\right\}^{1/p}.$$

By stationarity $E|X_t|^p=\text{Const}<\infty$ for all t, and thus $\{EZ_k^p\}^{1/p}\leq\text{Const}\,k^{-1}$ so that $\sum_{k=1}^{\infty}EZ_k^p<\infty$ from which it follows by the Borel-Cantelli lemma that $Z_k\to 0$ a.s.

The second step is to show that, since

$$\frac{1}{k}\int_0^k X_t\,\mathrm{d}t=\int_{-\infty}^{\infty}\frac{\mathrm{e}^{\mathrm{i}k\lambda}-1}{\mathrm{i}k\lambda}\,\mathrm{d}W(\lambda),$$

it suffices to show

$$Y_k\triangleq\int_{|\lambda|\leq 1}\frac{\mathrm{e}^{\mathrm{i}k\lambda}-1}{\mathrm{i}k\lambda}\,\mathrm{d}W(\lambda) \xrightarrow[k]{} W\{0\} \quad \text{a.s.}$$

since the remainder $R_k=\int_{|\lambda|>1}(\mathrm{e}^{\mathrm{i}k\lambda}-1)(\mathrm{i}k\lambda)^{-1}\,\mathrm{d}W(\lambda)$ tends to 0 with k a.s. Indeed we have

$$\|R_k\|_\alpha^\alpha=\int_{|\lambda|>1}\left|\frac{\sin k\lambda/2}{k\lambda/2}\right|^\alpha \mathrm{d}\mu(\lambda)\leq\frac{\text{Const}}{k^\alpha}.$$

Now with $X=\int f\,\mathrm{d}W$, it follows from (10) that $\|\mathrm{Re}\,X\|_\alpha^\alpha = \int_{-\infty}^{\infty}\int_{-\pi}^{\pi}|\mathrm{Re}\,f(\lambda)\,\mathrm{e}^{\mathrm{i}\theta}|^\alpha\,\mathrm{d}\nu(\lambda,\theta)$. The expression for $\|\mathrm{Im}\,X\|_\alpha^\alpha$ is then obtained simply by replacing f by $-if$. It then follows that

$$\begin{aligned}
E|X|^p &\leq E|\mathrm{Re}\,X|^p + E|\mathrm{Im}\,X|^p = \mathrm{Const}\{\|\mathrm{Re}\,X\|_\alpha^p + \|\mathrm{Im}\,X\|_\alpha^p\} \\
&= \mathrm{Const}\left\{\left(\int_{-\infty}^{\infty}\int_{-\pi}^{\pi}|\mathrm{Re}\,f(\lambda)\,\mathrm{e}^{\mathrm{i}\theta}|^\alpha\,\mathrm{d}\nu(\lambda,\theta)\right)^{p/\alpha}\right. \\
&\quad \left. +\left(\int_{-\infty}^{\infty}\int_{-\pi}^{\pi}|\mathrm{Re} - if(\lambda)\,\mathrm{e}^{\mathrm{i}\theta}|^\alpha\,\mathrm{d}\nu(\lambda,\theta)\right)^{p/\alpha}\right\} \\
&\leq \mathrm{Const}\|f\|_\alpha^p = \mathrm{Const}\|X\|_\alpha^p.
\end{aligned} \tag{12}$$

Thus for each $\varepsilon > 0$ and $1 < p < \alpha$,

$$P(|R_k| > \varepsilon) \leq \frac{E|R_k|^p}{\varepsilon^p} \leq \mathrm{Const}\,\frac{\|R_k\|_\alpha^p}{\varepsilon^p} \leq \frac{\mathrm{Const}}{\varepsilon^p k^p}$$

and $R_k \to 0$ a.s. follows from Borel-Cantelli.

The third step is to show that it suffices to establish a.s. convergence of Y_n along the subsequence $k=2^n$ since $\sup_{2^n < k \leq 2^{n+1}}|Y_k - Y_{2^n}| \to 0$ a.s. The fourth step is to show that

$$Y_{2^n} - \int_{|\lambda| < 2^n} \mathrm{d}W(\lambda) \underset{n}{\to} 0 \quad \text{a.s.}$$

and the final fifth step that

$$\int_{|\lambda| < 2^n} \mathrm{d}W(\lambda) \underset{n}{\to} W\{0\} \quad \text{a.s.}$$

These steps are established by adjusting Gaposhkin's arguments in ways similar to those exhibited in steps one and two—and need not be shown here.

We finally show that $W\{0\}=0$ if and only if $\|W\{0\}\|_\alpha = \mu\{0\} = 0$. This follows from (12) and

$$\begin{aligned}
\|X\|_\alpha &\leq \|\mathrm{Re}\,X\|_\alpha + \|\mathrm{Im}\,X\|_\alpha = \mathrm{Const}\{(E|\mathrm{Re}\,X|^p)^{1/p} + (E|\mathrm{Im}\,X|^p)^{1/p} \\
&\leq \mathrm{Const}(E|X|^p)^{1/p}. \qquad \square
\end{aligned}$$

6. Doubly stationary processes

We introduce in this section a new class of stationary SαS processes which we term *doubly stationary*. They are, loosely, those SαS processes whose spectral representations are themselves stationary.

To be more precise, let (E, Σ, μ) be an arbitrary (finite or infinite) measure space and let $\{f_t\colon t\in G\}$ be a collection of measurable functions on E. G is in general some group—for the purposes of this paper, we take it to be Z or $\mathbb{R}$. Call $\{f_t\}$ *stationary* if the μ-distribution of the vector $(f_{t_1+s},\ldots,f_{t_n+s})$ is independent of $s\in G$ for each fixed choice of n and $t_j\in G$. A SαS process will be called *doubly stationary* if it has the same distribution as some process $\{X_t=\int_E f_t(\lambda)\,\mathrm{d}z(\lambda)\colon t\in G\}$ where $\{f_t\}\subseteq L_\alpha(E,\Sigma,\mu)$ is stationary and Z is the canonical independently scattered random measure on (E,Σ,μ). It is clear by checking characteristic functions that doubly stationary SαS processes are also (strictly) stationary. Example (iv) below shows the converse does not hold.

For stationary $\{f_t\}$ we may find, just as in the case of a stationary process, a group of measure-preserving set transformations $\{T_t\}$ of $\Sigma=\sigma\{f_t\}$ such that $f_t=T_tf_0$. (We also denote by T_t the induced map on measurable functions.) Conversely, any group of measure-preserving set transformations defines stationary functions $\{T_tf_0\}$ for arbitrary measurable f_0. Thus a SαS process is doubly stationary if and only if it has a representation as in (2) of Section 1, where the group $\{U_t\}$ is induced by such a group $\{T_t\}$. This equivalent definition will be more useful for us, if not as picturesque.

Examples. (i) *Every mean-zero stationary Gaussian process is doubly stationary.* To see this, let $\{X_t\}$ be a mean-zero stationary Gaussian process on $(\Omega,\mathcal{F},P)$ and let Z be the canonical independently scattered Gaussian measure on $(E,\Sigma,\mu)\triangleq(\Omega,\mathcal{F},P)$. Then $\{Y_t\triangleq\int_\Omega X_t(\omega)\,\mathrm{d}Z(\omega)\}$ is seen (by checking characteristic functions) to have the same distribution as $\{X_t\}$. Hence $\{X_t\}$ is doubly stationary.

(ii) *Every stationary sub-Gaussian process is doubly stationary.* Let $\{X_t\}$ be α-sub-Gaussian on $(\Omega,\mathcal{F},P)$, represented as $X_t=A^{1/2}G_t$ as in Section 4. As seen in the proof of Theorem 3 in Section 4, $\{X_t\}$ is distributed as $\{Y_t\triangleq\int_\Omega cG_t(\omega)\,\mathrm{d}Z(\omega)\}$ where Z is the canonical independently scattered SαS random measure on $(\Omega,\mathcal{F},P)$ and c is a constant depending on α. $\{G_t\}$ is stationary since $\{X_t\}$ is, and thus $\{X_t\}$ is doubly stationary.

(iii) *All* SαS *moving averages are doubly stationary.* In this case, the group $\{T_t\}$ is the translation group on (G, Haar measure).

(iv) *There exists a stationary* SαS *process, continuous in probability, which is not doubly stationary.* For simplicity we take $\alpha=1$, although this example may be altered easily to work for each $\alpha\in(0,2)$. Define $U_t\colon L^1[0,1]\to L^1[0,1]$ for real t by $(U_tf)(x)=2^tx^{2^t-1}f(x^{2^t})$. It is easily checked that $\{U_t\}$ is a strongly continuous group of linear isometries, so that $\{X_t\triangleq\int_0^1U_t1_{[0,1]}\,\mathrm{d}Z\}$ is a stationary SαS process continuous in probability. Here, Z is Cauchy motion on $[0,1]$ (the canonical S1S independent increments process on $[0,1]$). We claim that $\{X_t\}$ is not doubly stationary.

For, if $\{X_t\}$ were doubly stationary, we could find a measure space (E,Σ,μ), a group of measure-preserving set maps $T_t\colon\Sigma\to\Sigma$ and a function $\phi\in L^\alpha(\mu)$ such $t\mapsto T_t\phi$ is a spectral representation for $\{X_t\}$. Since $t\mapsto U_t1_{[0,1]}$ is also a spectral representation for $\{X_t\}$ we must have that $\|\Sigma\, \lambda_j U_{t_j} f_0\|_{L^\alpha[0,1]} = \|\Sigma\, \lambda_j T_{t_j}\phi\|_{L^\alpha(\mu)}$ for all choices of λ_j and t_j. Hence the map $U_t 1_{[0,1]} \mapsto T_t\phi$ extends to a linear isometry of $\overline{\mathrm{sp}}\{U_t 1_{[0,1]}\}_{L^\alpha[0,1]}$ onto $\overline{\mathrm{sp}}\{T_t\phi\}_{L^\alpha(\mu)}$. This isometry in fact extends to *all* of $L^\alpha[0,1]$ by [11, Corollary 4.3], since $U_0 1_{[0,1]} = 1_{[0,1]}$ and $U_1 1_{[0,1]}(x) = 2x$ are both in $\overline{\mathrm{sp}}\{U_t 1_{[0,1]}\}$. Call this extension M. Again by [11, Corollary 4.3], M has the form $(Mf)(x) = h(x)(Sf)(x)$ where S is induced by a regular set isomorphism of ($\mathscr{B}$, Lebesgue) to (Σ, μ). Since $MU_t 1_{[0,1]} = T_t\phi$ we must have that, calling $\mathrm{id}(x) = x$,

$$T_t\phi = MU_t 1_{[0,1]} = M(2^t \mathrm{id}^{2^t-1}) = hS(2^t \mathrm{id}^{2^t-1}) = 2^t h[S(\mathrm{id})]^{2^t-1}.$$

Since $0 \leq \mathrm{id} < 1$ a.e., we have that $0 \leq S(\mathrm{id}) < 1$ a.e. $[\mu]$. If $0 \leq x < 1$ we have that $2^t x^{2^t-1} \to 0$ as $t \to \infty$. But $T_t\phi$ must be equidistributed for all t (since T_t is measure-preserving), and $2^t h[S(\mathrm{id})]^{2^t-1}$ by the above is not, since by choosing t large enough we may for any $\varepsilon > 0$ force $\mu\{|2^t h[S(\mathrm{id})]^{2^t-1}| < \varepsilon\}$ as close to $\mu(E)$ as desired. Therefore $\{X_t\}$ cannot be doubly stationary. $\square$

Remark. In view of the representation (2) and the fact that groups of isometries on L^α for $\alpha \neq 2$ are determined (essentially) by groups of transformations on the underlying measure space (see [16] or [11] for more details), it is natural to expect that many stationary SαS processes can be shown to be doubly stationary by "appropriately altering" the measure space upon which $\{U_t\}$ is defined.

We now turn to the ergodic properties of doubly stationary processes. For the remainder of this section, we assume that $\{X_t\}$ is a doubly stationary SαS process with spectral representation $t \mapsto T_t\phi$ where $\{T_t\}$ is a strongly measurable group induced by a group of measure-preserving set transformations on the arbitrary measure space (E, Σ, μ), and $\phi \in L^\alpha(\mu)$. We also assume WLOG that $\Sigma = \sigma\{T_t\phi\}$. Denote by $\mathscr{I}$ the invariant σ-field of $\{T_t\}$, $\mathscr{I} = \{A \in \Sigma\colon T_t A = A \text{ for all } t\}$.

The first result gives a sufficient condition for metric transivity. Note that condition (13) below on our "shift" $\{T_t\}$ of Σ and condition (4) of Section 2 on the shift in $(\Omega, \mathscr{F}, P)$ are of a fundamentally different nature—(13) is a kind of "asymptotic disjointness" condition, while (4) is a kind of asymptotic *independence* condition. This should not be too surprising, however, since it is known (see [20]) that two jointly SαS r.v.'s are independent if and only if their spectral representatives have disjoint support.

Theorem 6. *$\{X_t\}$ is metrically transitive if for all sets A, $B \in \Sigma$ of finite μ-measure*

$$\lim_{T \to \infty} \frac{1}{T} \int_0^T \mu(A \cap T_t B)\, \mathrm{d}t = 0. \tag{13}$$

Condition (13) guarantees that $\mu(E) = \infty$, for otherwise (13) would be false for $A = B = E$.

Proof. We first claim that it is enough to verify (5) and (6) of Theorem 1 for all simple functions g. The following inequalities are valid for real x and y:

$$|x|^\alpha - |y|^\alpha \le |x+y|^\alpha \le |x|^\alpha + |y|^\alpha \quad \text{for } 0<\alpha<1,$$

$$|x|^\alpha - |y|^\alpha - \alpha|x+y|^{\alpha-1}|y| \le |x+y|^\alpha \le |x|^\alpha + |y|^\alpha + \alpha|x|^{\alpha-1}|y| \quad \text{for } 1\le\alpha\le 2.$$

The first is well-known and the second follows from [18]. Call $x = T_t g - g$, $y = g - h + T_t(h-g)$ and $z = x + y = T_t h - h$, and integrate the inequalities (using Hölder in the second) to obtain

$$\|x\|_\alpha^\alpha - \|y\|_\alpha^\alpha \le \|z\|_\alpha^\alpha \le \|x\|_\alpha^\alpha + \|y\|_\alpha^\alpha, \quad 0<\alpha<1,$$

$$\|x\|_\alpha^\alpha - \|y\|_\alpha^\alpha - \alpha\|z\|_\alpha^{\alpha-1}\|y\|_\alpha \le \|z\|_\alpha^\alpha \le \|x\|_\alpha^\alpha + \|y\|_\alpha^\alpha + \alpha\|x\|_\alpha^{\alpha-1}\|y\|_\alpha,$$

$$1\le\alpha\le 2.$$

Now note that for arbitrary fixed h, $\|y\|_\alpha$ can be made uniformly small in t by choosing g simple with $\|h-g\|_\alpha$ small, and that $\|z\|_\alpha$ is uniformly bounded in t. These observations coupled with the inequalities above show that if (5) holds for all simple functions g, then it holds for all $h \in \overline{\mathrm{sp}}\{T_t\phi\}$ (and in fact for all $h \in L^\alpha(\mu)$). Squaring the inequalities above shows that the same thing can be said for condition (6). The claim is therefore true.

Now let $h = \sum_{j=0}^n c_j 1_{A_j} \in L^\alpha(\mu)$, with $\{A_j\}$ a partition of E and $A_0 = \{h=0\}$ (and of course $c_0 = 0$). Then $\mu(A_j) < \infty$ for $j \ge 1$ and $\mu(A_0) = \infty$. Call $A_{ij}(t) = T_t A_i \cap A_j$. Then $T_t h - h = \sum c_i 1_{T_t A_i} - \sum c_j 1_{A_j} = \sum_{i,j=0}^n (c_i - c_j) 1_{A_{ij}(t)}$ where $\{A_{ij}(t)\}$ partitions E, and we have

$$\|T_t h - h\|_\alpha^\alpha = \sum_{i,j=0}^n |c_i - c_j|^\alpha \mu(A_{ij}(t)). \tag{14}$$

Condition (13) guarantees that

$$\lim_{T\to\infty} T^{-1}\int_0^T \|T_t h - h\|_\alpha^\alpha \, \mathrm{d}t$$

$$= \lim_{T\to\infty} T^{-1}\int_0^T \left\{\sum_{i=1}^n |c_i|^\alpha \mu(A_{i0}(t)) + \sum_{j=1}^n |c_j|^\alpha \mu(A_{0j}(t))\right\} \mathrm{d}t. \tag{15}$$

It is not difficult to show that (13) also guarantees that $\lim_{T\to\infty} T^{-1}\int_0^T \mu(T_t B_1 \cap B_2)\,\mathrm{d}t$ is $\mu(B_1)$ [resp. $\mu(B_2)$] if B_1 and B_2^c have finite measure [resp. B_1^c and B_2 have finite measure]. Thus (15) shows that

$$\lim_{T\to\infty} T^{-1}\int_0^T \|T_t h - h\|_\alpha^\alpha = \sum |c_i|^\alpha \mu(A_i) + \sum |c_j|^\alpha \mu(A_j) = 2\|h\|_\alpha^\alpha$$

and so (5) holds.

To show (6) holds, note that from (14),

$$\|T_t h - h\|_\alpha^{2\alpha} = \sum_{i,j,k,l=0}^n |c_i - c_j|^\alpha |c_k - c_l|^\alpha \mu(A_{ij}(t))\mu(A_{kl}(t)).$$

Condition (13) applies to show

$$\begin{aligned}
&\lim_{T\to\infty} T^{-1}\int_0^T \|T_t h - h\|_\alpha^{2\alpha}\,\mathrm{d}t \\
&= \lim_{T\to\infty} T^{-1}\int_0^T \sum_{p,q=1}^{n} |c_p|^\alpha |c_q|^\alpha [\mu(A_{p0}(t))\mu(A_{q0}(t)) \\
&\quad + \mu(A_{p0}(t))\mu(A_{0q}(t)) + \mu(A_{0p}(t))\mu(A_{q0}(t)) \\
&\quad + \mu(A_{0p}(t))\mu(A_{0q}(t))]\,\mathrm{d}t. \qquad (16)
\end{aligned}$$

It can be shown with a little thought and a little bit more computation that Condition (13) also guarantees that $\lim_{T\to\infty} T^{-1}\int_0^T \mu(T_tB_1 \cap B_2)\mu(T_tB_3 \cap B_4)\,\mathrm{d}t$ is equal to $\mu(B_1)\mu(B_3)$ [resp. $\mu(B_1)\mu(B_4)$; $\mu(B_2)\mu(B_3)$; $\mu(B_2)\mu(B_4)$] if B_1, B_2^c, B_3, B_4^c [resp. B_1, B_2^c, B_3^c, B_4; B_1^c, B_2, B_3, B_4^c; B_1^c, B_2, B_3^c, B_4] have finite measure. Thus (16) shows that condition (6) holds, and $\{X_t\}$ must be metrically transitive.

We now give an analogous sufficient condition for mixing. As in the last result, the condition on $\{T_t\}$ here in (17) is of a fundamentally different nature than that on the shift of the process in (7) of Section 3.

Theorem 7. *$\{X_t\}$ is mixing if for all $A \in \sigma\{T_t\phi: t \le 0\}$, $B \in \sigma\{T_t\phi: t \ge 0\}$ of finite μ-measure,*

$$\lim_{t\to\infty} \mu(A \cap T_tB) = 0. \qquad (17)$$

Again, (17) guarantees that $\mu(E) = \infty$.

Proof. We will verify (9) of Theorem 2. Applying arguments similar to those in the proof of Theorem 6, we see that it is enough to have (9) for simple g and h in $L^\alpha(\mu)$. Let $A = \operatorname{supp}(g) \triangleq \{g \neq 0\}$, $B = \operatorname{supp}(h)$, and let g and h be bounded by M. Note that $\operatorname{supp}(T_th) = T_tB$ and that A and B are of finite measure. Since

$$\big|\|g + T_th\|_\alpha^\alpha - \|g\|_\alpha^\alpha - \|h\|_\alpha^\alpha\big| \le 2M^\alpha \mu(A \cap T_tB),$$

(17) shows that (9) holds and thus that $\{X_t\}$ is mixing.

We now look at laws of large numbers for doubly stationary processes. For simplicity we assume that $\alpha > 1$, so that $E|X_t| < \infty$ and we have (as in Section 5) that $\{X_t\}$ satisfies a SLLN if and only if it satisfies a weak LLN.

Note that for $X_t = \int_E T_t\phi\,\mathrm{d}Z$, $T^{-1}\int_0^T X_t\,\mathrm{d}t = \int_E (T^{-1}\int_0^T T_t\phi\,\mathrm{d}t)\,\mathrm{d}Z$, the change of integration being justified as in [4, Theorem 4.6]. But $T^{-1}\int_0^T T_t\phi\,\mathrm{d}t$ converges in L^α to $\bar{E}(\phi\,|\,\mathscr{I})$ by the mean ergodic theorem (see Theorem C of the Appendix). So by the definition of the stochastic integral, $T^{-1}\int_0^T X_t\,\mathrm{d}t$ converges in probability, and hence a.s., to $\int_E \bar{E}(\phi\,|\,\mathscr{I})\,\mathrm{d}Z$. This proves

Theorem 8. *Let $1 < \alpha < 2$. Then as $T \to \infty$, $T^{-1}\int_0^T X_t\,\mathrm{d}t$ converges a.s. and in probability to a random variable distributed as $\int_E \bar{E}(\phi\,|\,\mathscr{I})\,\mathrm{d}Z$. Thus $\{X_t\}$ satisfies the SLLN if and only if $\bar{E}(\phi\,|\,\mathscr{I}) = 0$ a.e. $[\mu]$.*

It follows, for $1 < \alpha < 2$, that *a necessary condition for metric transitivity of* $\{X_t\}$ *is* $\bar{E}(\phi \mid \mathscr{I}) = 0$; and that *if every set of* $\mathscr{I}$ *has either zero or infinite* μ*-measure, then* $\{X_t\}$ *satisfies the* SLLN.

These results allow us to construct examples showing that neither of the conditions "ergodicity of $\{T_t\}$" or "metric transitivity of $\{X_t\}$" implies the other. To mention one such example, let $1 < \alpha < 2$, $\{T_t\}$ be translation by $t \pmod 1$ on $[0, 1]$, $\phi = 1_{[0,1/2)} - 1_{[1/2,1)}$, and $X_t = \int_0^1 T_t\phi \, \mathrm{d}Z$. Then (i) $\{T_t\}$ *is ergodic* (ii) $\{X_t\}$ *satisfies the* SLLN, and (iii) $\{X_t\}$ *is not metrically transitive*. (i) is obvious, and (ii) follows from (i) and Theorem 8, since $\bar{F}(\phi \mid \mathscr{I}) = E(\phi \mid \mathscr{I}) = E\phi = 0$. To verify (iii), we note that

$$\lim_{T \to \infty} \frac{1}{T} \int_0^T |T_t\phi - \phi|^\alpha \, \mathrm{d}t = \int_0^1 |T_t\phi - \phi|^\alpha \, \mathrm{d}t$$

$$= \int_0^{1/2} 2^{\alpha+1} t \, \mathrm{d}t + \int_{1/2}^1 2^{\alpha+1}(1-t) \, \mathrm{d}t$$

$$= 2^{\alpha-1} \neq 2 = 2\|\phi\|^\alpha$$

and thus Condition (5) in Theorem 1 does not hold.

Appendix

We collect here some facts from ergodic theory needed throughout the paper. Although we expect that nothing in this presentation is new, we can find no reference for Theorems B and C. We assume that all (continuous parameter) groups are strongly measurable in order to be able to define the appropriate integrals (see [8, pp. 685–686]). We state Theorems A and C in the continuous case, but their discrete versions are also true.

Theorem A. *Let* $\{U_t\}$ *be a group of isometries on* $L^p(E, \Sigma, \mu)$, *where* (E, Σ, μ) *is an arbitrary measure space, and* $p > 1$. *Then for all* $\phi \in L^p(\mu)$, $T^{-1}\int_0^T U_t\phi \, \mathrm{d}t \to P\phi$ *as* $T \to \infty$, *where the convergence is a.e. and in* L^p, *and* P *is a projection operator onto* $M \triangleq \{f \in L^p(\mu): U_t f = f \text{ for all } t\}$.

Proof. The strong convergence follows from [8, p. 662] in the discrete case, and [8, p. 689] in the continuous case. That P is such a projection follows from [8, p. 662 and p. 688]. [1] shows that the convergence is also a.e.

When $\{U_t\}$ is induced by a group of measure-preserving set transformations $\{T_t\}$ on (E, Σ, μ), we can identify the limit operator P above. In the case $\mu(E) < \infty$, it is well known that P is the conditional expectation operator given the invariant sets of $\{T_t\}$. In the case $\mu(E) = \infty$, it is the appropriate generalization of such a conditional expectation, which we now describe.

Theorem B. *For (E, Σ, μ) an arbitrary measure space and Σ_0 a sub-σ-field of Σ, the orthogonal projection of $L^2(\Sigma)$ onto $L^2(\Sigma_0)$ has a unique extension by continuity to a positive contractive projection $\bar{E}(\cdot|\Sigma_0)$ of $L^p(\Sigma)$ onto $L^p(\Sigma_0)$ for $1 \le p \le 2$. $\bar{E}(f|\Sigma_0)$ is characterized as the unique Σ_0-measurable function in L^p satisfying $\int_A \bar{E}(f|\Sigma_0)\,\mathrm{d}\mu = \int_A f\,\mathrm{d}\mu$ for all $A \in \Sigma_0$ of finite measure.*

Proof. Denote by Q the orthogonal projection of $L^2(\Sigma)$ onto $L^2(\Sigma_0)$. For all $A \in \Sigma_0$ with finite measure and $f \in L^2(\Sigma)$, 1_A is orthogonal to $f - Qf$ and so

$$\int_A Qf\,\mathrm{d}\mu = \int_A f\,\mathrm{d}\mu. \tag{18}$$

This characterizes Q in the sense that Qf is the unique Σ_0-measurable function in L^2 which satisfies (18). Relation (18) also shows that Q is positive, i.e. $Qf \ge 0$ a.e. if $f \ge 0$ a.e., and $|Qf| \le Q|f|$. This, coupled with (18) and the fact that the support of any $f \in L^2$ is σ-finite, shows that $\|Qf\|_1 \le \|f\|_1$ for any $f \in L^1 \cap L^2$. Q now extends by continuity to a positive contractive projection $\bar{E}(\cdot|\Sigma_0)$ of $L^1(\Sigma)$ onto $L^1(\Sigma_0)$ satisfying (18) for all $f \in L^1$. The Riesz Convexity Theorem [8, p. 525] shows that the last statement is true with 1 replaced by p $(1 \le p \le 2)$.

It is clear that for $\mu(E) < \infty$, $\bar{E}(\cdot|\Sigma_0)$ is the standard conditional expectation operator $E(\cdot|\Sigma_0)$.

We can now state the ergodic theorem needed in Section 6.

Theorem C. *Let (E, Σ, μ) be arbitrary, the group $\{T_t\}$ be induced by a group of measure-preserving set transformations of Σ, and $\mathcal{I} = \{A \in \Sigma: T_t A = A$ for all $t\}$. For $\phi \in L^p(\mu)$,*

$$T^{-1}\int_0^T T_t\phi\,\mathrm{d}t \to \bar{E}(\phi|\mathcal{I})$$

as $T \to \infty$, where the convergence is a.e. and in L^p if (i) $1 < p \le 2$, or if (ii) $p = 1$ and $\mu(E) < \infty$.

Proof. Theorem A gives us that $T^{-1}\int_0^T T_t\,\mathrm{d}t$ converges to a projection P on $M = \{f \in L^p: T_t f = f$ for all $t\}$ in the appropriate senses if $1 < p \le 2$. For $p = 1$, the convergence follows from [8, p. 662 and p. 675] for the discrete case, and [8, p. 689 and p. 690] for the continuous case. It remains to identify P as $\bar{E}(\cdot|\mathcal{I})$.

P as an operator on L^2 must be a contraction, being the strong limit of contractions. In a Hilbert space there is but one contractive projection onto a given subspace, namely the orthogonal projection onto that subspace. Since it is easy to verify that $M = L^p(E, \mathcal{I}, \mu)$, we have that $P = \bar{E}(\cdot|\mathcal{I})$ on L^2, and hence that $P = \bar{E}(\cdot|\mathcal{I})$ on L^p by Theorem B. $\square$

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